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Spectral Representation of the Covariance Function of a Rotation-Scale-Reflection-Invariant Random Field

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ABSTRACT

A rotation-scale-reflection-invariant (RSRI) random field is defined in the strict sense as a spatial random field whose finite-dimensional distributions are invariant to rotations, rescaling, and reflections of the plane. In the weak sense, a RSRI random field is defined as one whose mean function is constant and whose covariance function depends only on the angle and minimal norm ratio between points on the plane. This paper derives the spectral representation of the RSRI covariance function. This is done by utilizing a connection between the covariance functions of RSRI random fields and homogeneous random fields. The results are used to construct an example of a valid RSRI covariance function.

Keywords: rotation-scale-reflection-invariant random fields, homogeneous random fields, covariance function, spectral representation

I. INTRODUCTION

A RSRI random field is defined in the strict sense as a spatial random field whose finite-dimensional distributions remain invariant to rotations, changes in scale, and reflections of the plane (Tejada, 2008). For a RSRI random field the mean function is constant and the covariance function depends only on the *angle* and *minimal norm ratio* between points on the plane. This paper derives the spectral representation of the covariance function of RSRI random fields. Such representation provides a characterization of the said covariance function, paving a way not only for determining whether a given function is a valid RSRI covariance function but also for constructing RSRI covariance functions. Furthermore, the spectral representation of the RSRI covariance function serves as a starting ground for the development of more general spectral theory (Gihman and Skorohod, 1974).

To be able to address the above-mentioned goal, we use a relationship between the covariance functions of RSRI random fields and homogeneous random fields. Rather than starting from fundamental principles, the spectral representation of the RSRI covariance function is derived from that of the *reflection-invariant homogeneous covariance function* found in Tejada (2008).

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II. ROTATION-SCALE-REFLECTION-INVARIANT RANDOM FIELDS

Let R_θ, $\theta \in (-\pi, \pi]$, denote the *rotation operator* on the Euclidean plane $\Re^2$ that rotates the point $\mathbf{s}$ into the point $R_\theta \mathbf{s}$ about the origin by an angle of radian measure θ. Denote by D_ϕ, $\phi > 0$, be the *dilation/contraction operator* on $\Re^2$ that dilates/contracts the point $\mathbf{s}$ into the point $D_\phi \mathbf{s} \equiv \phi \mathbf{s}$, where $\equiv$ denotes the assignment of a notation. Furthermore, let Q_1 be the *reflection operator* on $\Re^2$ that takes the point (s_1, s_2) over the horizontal axis to the point $Q_1(s_1, s_2) \equiv (s_1, -s_2)$, and Q_0 be the *identity transformation* that maps a point to itself, i.e., $Q_0(s_1, s_2) \equiv (s_1, s_2)$. Finally, define the *rotation-scale-reflection operator* by the composition $g_{\theta,\phi,\delta}\mathbf{s} \equiv R_\theta D_\phi Q_\delta \mathbf{s}$ which transforms the point $\mathbf{s}$ in $\Re^2$ via first a possible reflection, then a dilation/contraction, and then a rotation. The operator $g_{\theta,\phi,\delta}$ can be viewed as the affine matrix given by

$$g_{\theta,\phi,\delta} = \begin{bmatrix} \phi\cos\theta & (-1)^{\delta-1}\phi\sin\theta \\ \phi\sin\theta & (-1)^{\delta}\phi\cos\theta \end{bmatrix}$$

so that $g_{\theta,\phi,\delta}\mathbf{s}$ is a matrix product.

Let $\Re_0^2$ denote the Euclidean plane excluding the origin. One reason why the coordinate origin is excluded from the parameter space $\Re_0^2$ is that it stands as the vertex of the angles defined below and so it cannot be a leg of any angle. Moreover, the origin has zero length making the reciprocal of the minimal norm ratio between any arbitrary point and the origin undefined.

A random field $\{X(\mathbf{s}) : \mathbf{s} \in \Re_0^2\}$ is said to be *strictly RSRI* if it satisfies

$$\{X(g_{\theta,\phi,\delta}\mathbf{s}) : \mathbf{s} \in \Re_0^2\} \stackrel{d}{=} \{X(\mathbf{s}) : \mathbf{s} \in \Re_0^2\}$$

for all $\theta \in (-\pi, \pi]$, $\phi > 0$, and $\delta = 0,1$ (Tejada, 2008), where $\stackrel{d}{=}$ denotes stochastic equality which means that its finite-dimensional distributions satisfy

$$F_{g_{\theta,\phi,\delta}\{\mathbf{s}_1,\ldots,\mathbf{s}_n\}}(x_1,\ldots,x_n) = F_{\mathbf{s}_1,\ldots,\mathbf{s}_n}(x_1,\ldots,x_n)$$

for any $n = 1,2,\ldots$, any choice of spatial locations $\mathbf{s}_1,\ldots,\mathbf{s}_n \in \Re_0^2$, and any set of real numbers $x_1,\ldots,x_n$, where $g_{\theta,\phi,\delta}\{\mathbf{s}_1,\ldots,\mathbf{s}_n\} \equiv \{g_{\theta,\phi,\delta}\mathbf{s}_1,\ldots,g_{\theta,\phi,\delta}\mathbf{s}_n\}$.

For any point $\mathbf{s} \in \Re_0^2$, denote by $\theta_\mathbf{s} \in (-\pi, \pi]$ the radian measure of the angle $\mathbf{s}$ makes with the positive horizontal axis. For any pair $\mathbf{s}, \mathbf{t} \in \Re_0^2$, let $\theta_{\mathbf{s},\mathbf{t}}$ denote the *ordinary radian measure* of the angle between $\mathbf{s}$ and $\mathbf{t}$ with the origin as vertex so that

$$\theta_{\mathbf{s},\mathbf{t}} = |\mathrm{mod}(\theta_\mathbf{t} - \theta_\mathbf{s})| = |\mathrm{mod}(\theta_\mathbf{s} - \theta_\mathbf{t})|, \tag{2.1}$$

where "mod" returns the *minimal residue* [Weisstein, 1999-2008]. For an angle $\alpha = \theta + 2\pi z$ for some integer z and $\theta \in (-\pi, \pi]$, $\text{mod}(\alpha) = \theta$. Finally, let

$$\phi_{\mathbf{s},\mathbf{t}} \equiv \min\left\{\frac{\|\mathbf{s}\|}{\|\mathbf{t}\|}, \frac{\|\mathbf{t}\|}{\|\mathbf{s}\|}\right\} \tag{2.2}$$

be the *minimal norm ratio* between $\mathbf{s}$ and $\mathbf{t}$, where $\|\mathbf{s}\| \equiv \sqrt{s_1^2 + s_2^2}$ is the ordinary Euclidean norm.

For a strictly RSRI random field, the mean function is constant and the covariance between any two elements of the random field depends only on the angle measure and minimal norm ratio between their locations (Tejada, 2008). These findings form the idea behind the definition of a weakly RSRI random field.

Denote by L_2 the class of random variables that have finite squared expectations. An L_2 random field $\mathbf{X} \equiv \{X(\mathbf{s}) : \mathbf{s} \in \Re_0^2\}$ is defined to be *weakly RSRI* if for every $\mathbf{s} \in \Re_0^2$,

$$E[X(\mathbf{s})] = \mu$$

for some real number μ, and for any pair $\mathbf{s}, \mathbf{t} \in \Re_0^2$,

$$Cov(X(\mathbf{s}), X(\mathbf{t})) = B(\theta_{\mathbf{s},\mathbf{t}}, \phi_{\mathbf{s},\mathbf{t}}) \tag{2.3}$$

for some function B, called the *(RSRI) covariance-defining function* of $\mathbf{X}$ (Tejada, 2008). In other words, for a weakly RSRI random field the mean function is constant and the covariance between any two random variables $X(\mathbf{s})$ and $X(\mathbf{t})$, $\mathbf{s}, \mathbf{t} \in \Re_0^2$, does not depend directly on the absolute locations $\mathbf{s}$ and $\mathbf{t}$ but rather on their positions relative to each other and to the origin as provided jointly by the angle measure $\theta_{\mathbf{s},\mathbf{t}}$ and minimal norm ratio $\phi_{\mathbf{s},\mathbf{t}}$. It should be mentioned that any result regarding the covariance-defining function B amounts essentially to the same for the corresponding covariance function.

By definition, the class of weakly RSRI random fields contains the class of strictly RSRI random fields. That is, a strictly RSRI random field is necessarily a weakly RSRI random field. In view of covariance theory (Yaglom, 1986) which encompasses the objective of this paper, we restrict attention to weakly RSRI random fields. Hence, from this point onwards the term RSRI random field shall be used to refer to a weakly RSRI random field, unless specified otherwise. It is also assumed that all random fields discussed in the foregoing are L_2 random fields so that the relevant moments exist.

An example of a phenomenon that can be modeled by a RSRI random field is the fading error in measuring *radio propagation path loss* (Rappaport, 2002). It can be postulated that the covariance between the fading errors at two different locations is a function of their positions relative to each other and to the transmitter's location. The said relative positions can be captured simultaneously by the measure of the angle that the two locations form with the transmitter and the smaller ratio between the distances of the two locations from the transmitter.

III. MAIN RESULTS

This section tackles a correspondence between the covariance-defining functions of RSRI random fields and homogeneous random fields. Since the theory of homogeneous random fields is well established, the said correspondence can be used to develop some parts of the covariance theory of RSRI random fields. This approach proves to be relatively simpler than starting from basic principles and can be used to derive the spectral representation of RSRI covariance-defining functions. Such characterization leads to a way of constructing valid RSRI covariance-defining functions. Moreover, the spectral analysis of random fields sometimes provides insights that are not accessible through spatial-domain analysis.

3.1 Connection with homogeneous random fields

A *(weakly) homogeneous random field* say $\mathbf{Y} \equiv \{Y(\mathbf{p}) : \mathbf{p} \in \Re^2\}$ on the plane has a mean function that is constant and a covariance function that depends on the differences between points in $\Re^2$, that is, for all $\mathbf{p}, \mathbf{q} \in \Re^2$ we have $Cov(Y(\mathbf{p}), Y(\mathbf{q})) = H(\mathbf{p} - \mathbf{q})$ for some function H (Adler, 1981). If $\mathbf{Y}$ is also reflection-invariant then the function H satisfies $H(r_1, r_2) = H(|r_1|, |r_2|)$ for all $(r_1, r_2) \in \Re^2$ which allows H to be specified completely in $[0, \infty)^2$. Upon consolidation, we see that a random field on $\Re^2$ is *(weakly) reflection-invariant homogeneous (RIH)* if and only if its covariance function C satisfies

$$Cov(Y(\mathbf{p}), Y(\mathbf{q})) = H(|p_1 - q_1|, |p_2 - q_2|) \tag{3.1}$$

for all $\mathbf{p} \equiv (p_1, p_2)$ and $\mathbf{q} \equiv (q_1, q_2)$ in $\Re^2$. To facilitate the foregoing discussions, the function H is referred to as the *covariance-defining function* of $\mathbf{Y}$.

Let $\Re$ be the set of real numbers, ln be the natural logarithm function, and e be Euler's constant. The *log-polar transformation*, denoted as ρ, is a function with domain $\Re_0^2$ and range $\Re \times (-\pi, \pi]$ that maps a point $\mathbf{s} \in \Re_0^2$ with Cartesian coordinates (s_1, s_2) to the point

$$\rho(\mathbf{s}) \equiv (\ln\|\mathbf{s}\|, \theta_{\mathbf{s}})$$

in $\Re \times (-\pi, \pi]$ (Peters, et al., 1996). The log-polar transformation is bijective so that its inverse exists and is given by

$$\rho^{-1}(\mathbf{p}) \equiv (e^{p_1} \cos p_2, e^{p_1} \sin p_2)$$

for $\mathbf{p} \equiv (p_1, p_2)$ in $\Re \times (-\pi, \pi]$. Observe that e^{p_1} and p_2 are the polar coordinates of $\rho^{-1}(\mathbf{p})$ which means that its norm is

$$\|\rho^{-1}(\mathbf{p})\| = e^{p_1} \tag{3.2}$$

and its angle of inclination is

$$\theta_{\rho^{-1}(\mathbf{p})} = \mathrm{mod}(p_2). \tag{3.3}$$

The following proposition establishes a connection between the covariance functions of RSRI random fields and those of RIH random fields on $\Re \times (-\pi, \pi]$. Such connection is made possible by the log-polar transformation.

Proposition 3.1: Let $B : [0,\pi] \times (0,1] \to \Re$ and $H : [0,\infty) \times [0,\pi] \mapsto \Re$ be functions such that

$$B(\theta, \phi) = H(-\ln\phi, \theta) \tag{3.4}$$

for all $(\theta, \phi) \in [0,\pi] \times (0,1]$, or equivalently,

$$H(r_1, r_2) = B(r_2, e^{-r_1}) \tag{3.5}$$

for all $(r_1, r_2) \in [0,\infty) \times [0,\pi]$. Then B is the covariance-defining function of some RSRI random field if and only if H is the covariance-defining function of some RIH random field on $\Re \times [0,\pi]$.

Proof: Suppose first that B is the covariance-defining function of some RSRI random field $\mathbf{X} \equiv \{X(\mathbf{s}) : \mathbf{s} \in \Re_0^2\}$, that is, the covariance function $C_{\mathbf{X}}(\mathbf{s}, \mathbf{t}) \equiv Cov(X(\mathbf{s}), X(\mathbf{t}))$ of $\mathbf{X}$ is such that $C_{\mathbf{X}}(\mathbf{s}, \mathbf{t}) = B(\theta_{\mathbf{s},\mathbf{t}}, \phi_{\mathbf{s},\mathbf{t}})$ for all $\mathbf{s}, \mathbf{t} \in \Re_0^2$. Let $H : [0,\infty) \times [0,\pi] \mapsto \Re$ be the function in (3.5). We show that H is the covariance-defining function of some RIH random field on $\Re \times [0,\pi]$. Let $\mathbf{p} \equiv (p_1, p_2)$ and $\mathbf{q} \equiv (q_1, q_2)$ be points in $\Re \times [0,\pi]$. Then from (3.5)

$$H(|p_1 - q_1|, |p_2 - q_2|) = B(|p_2 - q_2|, e^{-|p_1 - q_1|}). \tag{3.6}$$

Since p_2 and q_2 are in $[0,\pi]$ we have $\text{mod}(p_1) = p_1$ and $\text{mod}(p_2) = p_2$ which makes their absolute difference fall in $[0,\pi]$ making $|\text{mod}(p_2 - q_2)| = |p_2 - q_2|$. Hence,

$$\begin{aligned}
|p_2 - q_2| &= |\text{mod}(p_2 - q_2)| \\
&= |\text{mod}(\text{mod}(p_2) - \text{mod}(q_2))| \\
&= \left|\text{mod}\left(\theta_{\rho^{-1}(\mathbf{p})} - \theta_{\rho^{-1}(\mathbf{q})}\right)\right| \\
&= \theta_{\rho^{-1}(\mathbf{p}), \rho^{-1}(\mathbf{q})}
\end{aligned}$$

using (3.3) and (2.1). Also, (3.2) and (2.2) gives

$$\begin{aligned}
e^{-|p_1 - q_1|} &= \begin{cases} e^{q_1 - p_1} & \text{if } p_1 \geq q_1 \\ e^{p_1 - q_1} & \text{if } p_1 < q_1 \end{cases} \\
&= \min\left\{e^{p_1 - q_1}, e^{q_1 - p_1}\right\} \\
&= \min\left\{\frac{e^{p_1}}{e^{q_1}}, \frac{e^{q_1}}{e^{p_1}}\right\} \\
&= \min\left\{\frac{\|\rho^{-1}(\mathbf{p})\|}{\|\rho^{-1}(\mathbf{q})\|}, \frac{\|\rho^{-1}(\mathbf{q})\|}{\|\rho^{-1}(\mathbf{p})\|}\right\} \\
&= \phi_{\rho^{-1}(\mathbf{p}), \rho^{-1}(\mathbf{q})}.
\end{aligned}$$

From (3.6) and (2.3), we have

$$\begin{aligned} H(|p_1-q_1|,|p_2-q_2|) &= B\left(\theta_{\rho^{-1}(\mathbf{p}),\rho^{-1}(\mathbf{q})},\phi_{\rho^{-1}(\mathbf{p}),\rho^{-1}(\mathbf{q})}\right) \\ &= C_{\mathbf{X}}\left(\rho^{-1}(\mathbf{p}),\rho^{-1}(\mathbf{q})\right) \\ &= Cov\left[X\left(\rho^{-1}(\mathbf{p})\right),X\left(\rho^{-1}(\mathbf{q})\right)\right]. \end{aligned}$$

Let $Y(\mathbf{p})=X\left(\rho^{-1}(\mathbf{p})\right)$ and $Y(\mathbf{q})=X\left(\rho^{-1}(\mathbf{q})\right)$. Thus, we have

$$H(|p_1-q_1|,|p_2-q_2|)=Cov[Y(\mathbf{p}),Y(\mathbf{q})].$$

Hence, the covariance function of the random field $\mathbf{Y}\equiv\{Y(\mathbf{p}):\mathbf{p}\in\Re\times[0,\pi]\}$ is a function of $\mathbf{p},\mathbf{q}\in\Re\times[0,\pi]$ only through $(|p_1-q_1|,|p_2-q_2|)$. The random field $\mathbf{Y}$ has a constant mean function because $\mathbf{X}$ has which proves that $\mathbf{Y}$ is RIH with H as its covariance-defining function.

Conversely, suppose that H is the covariance-defining function of some RIH random field, say $\mathbf{Y}$, on $\Re\times[0,\pi]$. Then the covariance function of $\mathbf{Y}$ satisfies $C_{\mathbf{Y}}(\mathbf{p},\mathbf{q})=H(|p_1-q_1|,|p_2-q_2|)$ for all $\mathbf{p}\equiv(p_1,p_2)$ and $\mathbf{q}\equiv(q_1,q_2)$ in $\Re\times[0,\pi]$. Let B be the function in (3.4) for $(\theta,\phi)\in[0,\pi]\times(0,1]$. We show that B is a RSRI covariance-defining function. Denote by $\Re_U^2$ the upper Cartesian plane including the horizontal axis but excluding the origin, that is, $\Re_U^2\equiv\Re_0^2-(\Re\times\Re^-)$. Let $\mathbf{s}$ and $\mathbf{t}$ be a pair of arbitrary points in $\Re_U^2$. From (3.4), we have

$$B(\theta_{\mathbf{s},\mathbf{t}},\phi_{\mathbf{s},\mathbf{t}})=H(-\ln\phi_{\mathbf{s},\mathbf{t}};\theta_{\mathbf{s},\mathbf{t}}). \tag{3.7}$$

Now, the first argument of H can be derived as

$$\begin{aligned} -\ln\phi_{\mathbf{s},\mathbf{t}} &= -\ln\left(\min\left\{\frac{\|\mathbf{t}\|}{\|\mathbf{s}\|},\frac{\|\mathbf{s}\|}{\|\mathbf{t}\|}\right\}\right) \\ &= \ln\left(\frac{1}{\min\left\{\frac{\|\mathbf{t}\|}{\|\mathbf{s}\|},\frac{\|\mathbf{s}\|}{\|\mathbf{t}\|}\right\}}\right) \\ &= \ln\left(\max\left\{\frac{\|\mathbf{s}\|}{\|\mathbf{t}\|},\frac{\|\mathbf{t}\|}{\|\mathbf{s}\|}\right\}\right) \\ &= \begin{cases} \ln\frac{\|\mathbf{s}\|}{\|\mathbf{t}\|} & \text{if } \|\mathbf{s}\|\ge\|\mathbf{t}\| \\ \ln\frac{\|\mathbf{t}\|}{\|\mathbf{s}\|} & \text{if } \|\mathbf{s}\|<\|\mathbf{t}\| \end{cases} \end{aligned}$$

$$= \left| \ln \frac{\|\mathbf{s}\|}{\|\mathbf{t}\|} \right| .$$

Since $\mathbf{s},\mathbf{t} \in \Re_U^2$, the angles $\theta_{\mathbf{s}}$ and $\theta_{\mathbf{t}}$ are in $[0,\pi]$. This means that $|\theta_{\mathbf{s}} - \theta_{\mathbf{t}}|$ reduces to $\theta_{\mathbf{s},\mathbf{t}}$. Hence, from (3.7) and (3.1),

$$\begin{aligned} B(\theta_{\mathbf{s},\mathbf{t}}, \phi_{\mathbf{s},\mathbf{t}}) &= H(-\ln \phi_{\mathbf{s},\mathbf{t}}, \theta_{\mathbf{s},\mathbf{t}}) \\ &= H\left(\left| \ln \frac{\|\mathbf{s}\|}{\|\mathbf{t}\|} \right|, |\theta_{\mathbf{s}} - \theta_{\mathbf{t}}| \right) \\ &= H\left(\left| \ln\|\mathbf{s}\| - \ln\|\mathbf{t}\| \right|, |\theta_{\mathbf{s}} - \theta_{\mathbf{t}}| \right). \\ &= C_{\mathbf{Y}}\left((\ln\|\mathbf{s}\|, \theta_{\mathbf{s}}), (\ln\|\mathbf{t}\|, \theta_{\mathbf{t}}) \right) \\ &= C_{\mathbf{Y}}(\rho(\mathbf{s}), \rho(\mathbf{t})) \\ &= Cov[Y(\rho(\mathbf{s})), Y(\rho(\mathbf{t}))]. \end{aligned}$$

Let $X_U(\mathbf{s}) \equiv Y(\rho(\mathbf{s}))$ and $X_U(\mathbf{t}) \equiv Y(\rho(\mathbf{t}))$. We have

$$B(\theta_{\mathbf{s},\mathbf{t}}, \phi_{\mathbf{s},\mathbf{t}}) = Cov[X_U(\mathbf{s}), X_U(\mathbf{t})].$$

Since $\mathbf{Y}$ has a constant mean function then so does the random field $\mathbf{X}_U \equiv \{X_U(\mathbf{s}) : \mathbf{s} \in \Re_U^2\}$. Therefore, $\mathbf{X}_U$ is a RSRI random field and B is its covariance-defining function.

What remains to be seen is that B is the covariance-defining function of some RSRI random field on the entire parameter space $\Re_0^2$. For any pair $\mathbf{s},\mathbf{t} \in \Re_0^2$, there exists a pair $\mathbf{s}',\mathbf{t}' \in \Re_U^2$ such that $\theta_{\mathbf{s}',\mathbf{t}'} = \theta_{\mathbf{s},\mathbf{t}}$ and $\phi_{\mathbf{s}',\mathbf{t}'} = \phi_{\mathbf{s},\mathbf{t}}$, that is, the angle measure and minimal norm ratio between $\mathbf{s}'$ and $\mathbf{t}'$ coincide with those between $\mathbf{s}$ and $\mathbf{t}$. For instance, apply the rotation $R_{-\theta_{\mathbf{s}}}$ on $\mathbf{s}$ and $\mathbf{t}$ yielding the points $R_{-\theta_{\mathbf{s}}}\mathbf{s}$ and $R_{-\theta_{\mathbf{s}}}\mathbf{t}$, realizing that because of the rotation angle $-\theta_{\mathbf{s}}$, the point $R_{-\theta_{\mathbf{s}}}\mathbf{s}$ falls on the positive side of the horizontal axis. Next, if $R_{-\theta_{\mathbf{s}}}\mathbf{t}$ falls in $\Re_0^2 - \Re_U^2$, i.e., the lower half-plane, apply the reflection Q_1 so that the point $Q_1 R_{-\theta_{\mathbf{s}}}\mathbf{t}$ appears in the upper half-plane $\Re_U^2$. On the other hand, if $R_{-\theta_{\mathbf{s}}}\mathbf{t}$ is already in $\Re_U^2$ no reflection is made which is similar to applying the operator Q_0. Write $\mathbf{t}' \equiv Q_\delta R_{-\theta_{\mathbf{s}}}\mathbf{t}$, where δ is 0 or 1 depending on which will make $\mathbf{t}'$ a point in $\Re_U^2$. Note that the point $\mathbf{s}' \equiv Q_\delta R_{-\theta_{\mathbf{s}}}\mathbf{s}$ is still the same point $R_{-\theta_{\mathbf{s}}}\mathbf{s}$ that is on the positive horizontal axis whatever the value of δ may be. It is then easy to see that the angle between $\mathbf{s}'$ and $\mathbf{t}'$ is the angle of inclination of $\mathbf{t}'$ which has positive measure and is in fact equal to $\theta_{\mathbf{s},\mathbf{t}}$. Moreover, the minimal norm ratio $\phi_{\mathbf{s}',\mathbf{t}'}$ between $\mathbf{s}'$ and $\mathbf{t}'$ is the same as $\phi_{\mathbf{s},\mathbf{t}}$.

Now, let $\mathbf{X} \equiv \{X(\mathbf{s}) : \mathbf{s} \in \Re_0^2\}$ be a random field having a constant mean function and a covariance function $C_{\mathbf{X}}$ that is defined as $C_{\mathbf{X}}(\mathbf{s},\mathbf{t}) = C_{\mathbf{X}_U}(\mathbf{s}',\mathbf{t}')$ for all $\mathbf{s},\mathbf{t} \in \Re_0^2$, where $C_{\mathbf{X}_U}$

is the covariance function of $\mathbf{X}_U$. Since $(\theta_{\mathbf{s},\mathbf{t}},\phi_{\mathbf{s},\mathbf{t}})\in[0,\pi]\times(0,1]$, $B(\theta_{\mathbf{s},\mathbf{t}},\phi_{\mathbf{s},\mathbf{t}})$ is well-defined and can be expressed as

$$\begin{aligned} B(\theta_{\mathbf{s},\mathbf{t}},\phi_{\mathbf{s},\mathbf{t}}) &= B(\theta_{\mathbf{s}',\mathbf{t}'},\phi_{\mathbf{s}',\mathbf{t}'}) \\ &= C_{\mathbf{X}_U}(\mathbf{s}',\mathbf{t}') \\ &= C_{\mathbf{X}}(\mathbf{s},\mathbf{t}) \end{aligned}$$

for all $\mathbf{s},\mathbf{t}\in\Re_0^2$. Therefore, $\mathbf{X}$ is a RSRI random field with B as its covariance-defining function. □

3.2 Spectral representation of RSRI covariance-defining functions

Let S be a continuous subset of $\Re^2$. A random field $\{X(\mathbf{s}):\mathbf{s}\in S\}$ is said to be *mean-square continuous at the point* $\mathbf{s}_0\in S$ if for any sequence $\mathbf{s}_1,\mathbf{s}_2,\ldots$ in S converging to $\mathbf{s}_0$ we have $E\left[(X(\mathbf{s}_n)-X(\mathbf{s}_0))^2\right]\to 0$ as $n\to\infty$. If the random field is mean-square continuous at every $\mathbf{s}\in S$, then it is said to be *mean-square continuous on* S.

The following proposition gives the spectral representation of a RSRI covariance-defining function. The keys to this result are the correspondence between the covariance-defining functions of RSRI random fields and RIH random fields as given by Proposition 3.1 and the spectral representation of RIH random fields as derived in Tejada (2008). In the foregoing, integrals are to be taken in the Lebesgue sense unless stated otherwise.

Proposition 3.2: A function $B:[0,\pi]\times(0,1]\mapsto\Re$ is the covariance-defining function of some mean-square continuous RSRI random field if and only if it can be represented as

$$B(\theta,\phi)=\int_{[0,\infty)^2}\cos(u_1\ln\phi)\cos(\theta u_2)F(d(u_1,u_2)) \tag{3.8}$$

for all $(\theta,\phi)\in[0,\pi]\times(0,1]$, where F is a finite measure on the Borel sets of $[0,\infty)^2$.

Proof: The necessity is proved first. Let B be the covariance-defining function of some RSRI random field, say $\mathbf{X}\equiv\{X(\mathbf{s}):\mathbf{s}\in\Re_0^2\}$, that is continuous in mean square. For each pair $(r_1,r_2)\in[0,\infty)\times[0,\pi]$, define $H(r_1,r_2)\equiv B(r_2,e^{-r_1})$. By Proposition 3.1, H is the covariance-defining function of some RIH random field, say $\mathbf{Y}$, on $\Re\times[0,\pi]$. Since $\mathbf{X}$ is mean-square continuous then B is continuous at $(0,1)$ (Tejada, 2008). This in turn implies that H, by definition, is continuous at the origin which means that $\mathbf{Y}$ is mean-square-continuous (Adler, 1981). Hence, from Tejada (2008), the covariance-defining function H of $\mathbf{Y}$ can be expressed as

$$H(r_1,r_2)=\int_{[0,\infty)^2}\cos(r_1u_1)\cos(r_2u_2)F(d(u_1,u_2))$$

for all $(r_1,r_2)\in[0,\infty)^2$, where F is a finite measure on $[0,\infty)^2$. The relationship $H(r_1,r_2)=B(r_2,e^{-r_1})$ is equivalent to $B(\theta,\phi)=H(-\ln\phi,\theta)$ for all $(\theta,\phi)\in[0,\pi]\times(0,1]$. We can therefore express the RSRI covariance-defining function B as

$$B(\theta,\phi)=H(-\ln\phi,\theta)$$

$$= \int_{[0,\infty)^2} \cos((-\ln\phi)u_1)\cos(\theta u_2)F(d(u_1,u_2))$$
$$= \int_{[0,\infty)^2} \cos(u_1 \ln\phi)\cos(\theta u_2)F(d(u_1,u_2))$$

for all $(\theta,\phi) \in [0,\pi]\times(0,1]$, where the last equality is due to the cosine function being an even function.

Suppose that a function B satisfies (3.8) for all $(\theta,\phi) \in [0,\pi]\times(0,1]$, where F is a finite measure on $[0,\infty)^2$. It is to be shown that B is the covariance-defining function of some mean-square continuous RSRI random field. Defining once more $H(r_1,r_2) \equiv B(r_2,e^{-r_1})$ for $(r_1,r_2) \in [0,\infty)\times[0,\pi]$, we have from (3.8)

$$\begin{aligned} H(r_1,r_2) &= B(r_2,e^{-r_1}) \\ &= \int_{[0,\infty)^2} \cos(u_1 \ln(e^{-r_1}))\cos(r_2 u_2)F(d(u_1,u_2)) \\ &= \int_{[0,\infty)^2} \cos(u_1(-r_1))\cos(r_2 u_2)F(d(u_1,u_2)) \\ &= \int_{[0,\infty)^2} \cos(r_1 u_1)\cos(r_2 u_2)F(d(u_1,u_2)). \end{aligned}$$

Hence, from Tejada (2008), H is the covariance-defining function of some mean-square continuous RIH random field $\mathbf{Y}$ on $\Re\times[0,\pi]$. Noting once more that $B(\theta,\phi) = H(-\ln\phi,\theta)$ for all $(\theta,\phi) \in [0,\pi]\times(0,1]$, by Proposition 3.1, B is the covariance function of some RSRI random field $\mathbf{X}$. Since $\mathbf{Y}$ is mean-square continuous, its covariance-defining function H is continuous at the origin (Adler, 1981) implying that B is continuous at $(0,1)$. This leads to the conclusion that the RSRI random field $\mathbf{X}$ is mean-square continuous (Tejada, 2008). □

IV. AN APPLICATION

The spectral representation in Proposition 3.2 provides a necessary and sufficient condition for a given function to be a RSRI covariance-defining function. This characterization can be used to arrive at valid RSRI covariance-defining functions. All we need is a finite measure on $[0,\infty)^2$ which could be obtained by restricting any finite measure on $\Re^2$ to the Borel sets of $[0,\infty)^2$. Alternatively, any distribution function on $\Re^2$ that is bounded and monotonically increasing on $\Re^2$ may be used. However, the simplest way to go about using the spectral representation for the said purpose is to specify any function f on $\Re^2$ that is nonnegative and Lebesgue integrable. Equation (3.8) becomes

$$B(\theta,\phi) = \int_{[0,\infty)^2} \cos(u_1 \ln\phi)\cos(\theta u_2)f(u_1,u_2)d(u_1,u_2) \tag{4.1}$$

for $(\theta,\phi) \in [0,\pi]\times(0,1]$. If f is chosen to be Riemann integrable then we may interpret (4.1) in the much simpler Riemann sense since the cosine function is also Riemann integrable.

To illustrate the above-described approach, we provide a simple example. Suppose that f is the so-called "rectangular" function on $\Re^2$ given by

$$f(u_1,u_2) \equiv \begin{cases} \alpha, & (u_1,u_2) \in [0,\beta]^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha, \beta \geq 0$. Such spectral density represents a flat-response band pass which cuts off spatial frequencies outside of $[0,\beta]^2$. Note that f becomes the *uniform* density function on the square $[0,\beta]^2$ if β is positive and $\alpha = \frac{1}{\beta^2}$. It is not difficult to verify that f is a Riemann-integrable function. Evaluating (4.1) leads to the RSRI covariance-defining function

$$\begin{aligned} B(\theta,\phi) &= \int_{[0,\infty)^2} \cos(u_1 \ln\phi)\cos(\theta u_2) f(u_1,u_2) d(u_1,u_2) \\ &= \int_{[0,\beta]^2} \cos(u_1 \ln\phi)\cos(\theta u_2) \alpha d(u_1,u_2) \\ &= \alpha \int_{[0,\beta]^2} \cos(u_1 \ln\phi)\cos(\theta u_2) d(u_1,u_2) \end{aligned}$$

for all $(\theta,\phi) \in [0,\pi] \times (0,1]$. Since the integrand is continuous over $[0,\beta]^2$ we have from Fubini's Theorem (Billingsley, 1986)

$$\begin{aligned} B(\theta,\phi) &= \alpha \int_0^\beta \left(\int_0^\beta \cos(u_1 \ln\phi)\cos(\theta u_2) du_1 \right) du_2 \\ &= \alpha \int_0^\beta \cos(\theta u_2) \left(\int_0^\beta \cos(u_1 \ln\phi) du_1 \right) du_2 \\ &= \alpha \left(\int_0^\beta \cos(u_1 \ln\phi) du_1 \right) \left(\int_0^\beta \cos(\theta u_2) du_2 \right) \\ &= \alpha \left(\frac{\sin(u_1 \ln\phi)}{\ln\phi} \Bigg|_0^\beta \right) \left(\frac{\sin(\theta u_2)}{\theta} \Bigg|_0^\beta \right) \\ &= \alpha \frac{\sin(\beta \ln\phi)}{\ln\phi} \frac{\sin(\beta\theta)}{\theta} \\ &= \frac{\alpha \sin(\beta \ln\phi) \sin(\beta\theta)}{\theta \ln\phi} \end{aligned}$$

for all $(\theta,\phi) \in (0,\pi] \times (0,1)$. If we adhere to the usual convention of defining $\frac{\sin x}{x} \equiv 1$ (Maor, 1998) whenever $x = 0$, then B is well-defined for all $(\theta,\phi) \in [0,\pi] \times (0,1]$. Figure 4 shows the curve representing B, with α, β equal to one.

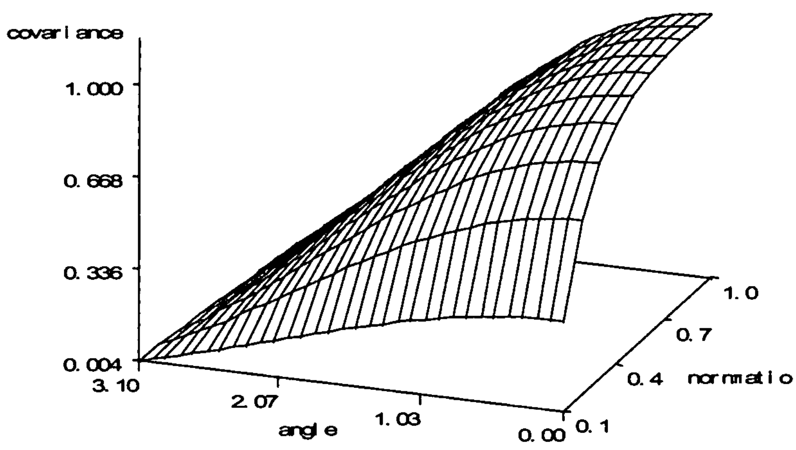

Figure 1: RSRI covariance-defining function $B(\theta,\phi) \equiv \dfrac{\sin(\ln\phi)\sin(\theta)}{\theta\log\phi}$.

V. CONCLUDING REMARKS

The spectral representation derived in this paper paves the way for the development of more general spectral theory regarding RSRI random fields. This includes the establishment of the spectral representation of the RSRI random field itself not just its covariance function. The results of this research also serve as a starting point in the derivation of inversion formulas, if they exist. Inversion formulas determine the spectral measure, spectral distribution function, and/or spectral density of the RSRI covariance function. Moreover, Fourier analyses of the RSRI covariance function could be subsequently performed as the spectral representation decomposes the covariance function into its harmonic components given by sinusoidal waves.

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